Phase theory and critical exponents for the Tomonaga-Luttinger model with harmonic confinement

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Abstract. A phase operator formulation for a recent model of interacting one-dimensional fermions in a harmonic trap is developed. The resulting theory is similar to the corresponding approach for the Luttinger model with open boundary conditions (OBC). However, in place of the spatial coordinate z, a dimensionless variable u defined on the unit circle appears as argument of the phase fields and u is non-linearly related to z. Furthermore, form factors appear which reflect the harmonic trap geometry. The theory is applied to calculate one-particle correlation functions. In a properly defined thermodynamic limit, bulk and boundary critical exponents are calculated for the static two-point correlation function and the dynamic local correlation function. The local spectral density is also considered. The critical exponents found are in agreement with those known for OBC with the exception of the boundary scaling exponent $\Delta_\perp$.

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1. Introduction

The concepts underlying the Luttinger model (LM) [1, 2, 3, 4] (for reviews cf. [5, 6, 7, 8]) find increasing application in the theory of quasi one-dimensional ultracold quantum gases. Using the harmonic fluid model of [9], properties of a one-dimensional trapped Bose gas were investigated in [10]. Related concepts were also used in the theory of interacting bosons in one-dimensional optical lattices [11, 12]. In [13], mass-composition separation as analogue of spin-charge separation in the LM was studied for a two-component neutral Fermi gas in a one-dimensional harmonic trap (cf. also [14]). A detailed description of the harmonic fluid approach to one-dimensional quantum gases is given in [15].

The present authors formulated a theory of interacting one-dimensional fermions in a harmonic atom trap [14, 17] based on bosonization in analogy to the LM. The theory was initiated by the prospect to create experimentally a highly degenerate Fermi gas of neutral atoms in a quasi one-dimensional harmonic trap. A possible route to achieve this aim is microtrap technology (cf.[18, 19, 20, 21, 22]). This will allow to study non-Fermi liquid behavior of interacting one-dimensional Fermi gases without the complications due to contacts and impurities.

The model in [14, 17], which can be termed Tomonaga-Luttinger model with harmonic confinement, lacks so far a phase operator formulation known for many models of one-dimensional interacting fermions. Probably best known for such a phase operator formulation is the Tomonaga-Luttinger model with periodic boundary conditions. A phase formulation for the corresponding model with open boundary conditions (OBC), which belongs to another universality class [23], is found in [24] and in particular in [25]. Related phase theories also exist for the case of interacting one-dimensional bosons [9, 15, 16].

In general, a phase operator approach to one-dimensional fermions comprises several features:

i) Extending the linearized dispersion near the Fermi energy with respect to fermionic creation and annihilation operators $\hat{c}^+_{k\sigma}$ and $\hat{c}_{k\sigma}$ to all values of k and adding an anomalous vacuum of filled negative energy states, Kronig's identity [26] can be used to transform the free Hamiltonian into a free bosonic form involving phonon-like density fluctuation operators $\hat{b}^+_{q\sigma}$ and $\hat{b}_{q\sigma}$. The index $\sigma = \pm 1$ accounts for two spin directions or two different components in the Fermi gas.

ii) The pair interactions between fermions must be describable in terms of bilinear products of density fluctuation operators.

iii) In the spinful case, an additional transformation to charge and spin degrees of freedom (or mass and composition degrees in the neutral case), denoted by $\nu = \pm 1$, is needed on the way to a diagonal Hamiltonian:

$$\hat{d}_{q\nu} \equiv \frac{1}{\sqrt{2}} \sum_{\sigma} \sigma^{\frac{1-\nu}{2}} \hat{b}_{q\sigma}, \quad \nu = \pm 1. \tag{1}$$

iv) In the absence of backscattering between different components, the Hamiltonian then separates into decoupled spin and charge (or mass and composition) parts which can be diagonalized by a Bogoliubov transformation. The final Hamiltonian $\tilde{H}$ for the excitations of the one-dimensional Fermi sea typically has the structure $\tilde{H} = \sum_{q\nu} v_{q\nu} \, |q| \, \hat{f}^+_{q\nu} \hat{f}_{q\nu}$ in terms of free phonon operators $\hat{f}^+$ and $\hat{f}$ and renormalized velocities $v_{q\nu}$. Zero modes [4] must be added to make the correspondence with the original fermionic Hamiltonian complete.

v) If the dependence of the velocities on q, i.e., that of the underlying coupling functions is neglected, hermitian phase fields $\hat{\Phi}_\nu(z)$ and dual fields $\hat{\Theta}_\nu(z)$ can be defined in terms of the $\hat{d}$-operators (or else in terms of the $\hat{f}$-operators) which contain zero modes and bring the Hamiltonian into the canonical form

$$\hat{H} = \sum_\nu \frac{v_\nu}{2} \int dz \left[\pi K_\nu \left(\partial_z \hat{\Theta}_\nu(z) \right)^2 + \frac{1}{\pi K_\nu} \left(\partial_z \hat{\Phi}_\nu(z) \right)^2 \right], \tag{2}$$

where K_ν are the central coupling constants. Backscattering between the two components $\hat{V}_\perp \propto \int dz \, \cos\left[\sqrt{8}\, \hat{\Phi}_{-1}(z)\right]$ destroys the simple quadratic form of the spin part [27, 28].

vi) An important step in any phase theory is the representation of fermion field operators in terms of the phase operators in order to calculate correlation functions easily, a procedure usually called "bosonization". The method goes back to [29] and [30] and is now well understood [31, 4]. For open boundaries, it is described in the works [24] and [25].

Here, we consider interacting one-dimensional fermions in a harmonic trap. Then the single particle wave functions are not simple combinations of plane waves. We, therefore, use a general bosonization prescription described in [32] which involves auxiliary Fermi fields $\hat{\psi}_{a\sigma}$ related to the original operators $\hat{c}_{l\sigma}$ (note that for the harmonic oscillator q is replaced by a discrete index $l = ..., -2, -1, 0, 1, 2, ...$) by

$$\hat{\psi}_{a\sigma}(u) \equiv \sum_{l=-\infty}^{\infty} e^{ilu} \, \hat{c}_{l\sigma} = \hat{\psi}_{a\sigma}(u + 2\pi). \tag{3}$$

These operators are defined on the unit circle as will be most other operators below. Bosonization is provided by [32, 33]

$$\hat{\psi}_{a\sigma}(u) = e^{iu\hat{N}_\sigma} \, e^{i\hat{\phi}^+_\sigma(u)} \, e^{i\hat{\phi}_\sigma(u)} \, \hat{U}_\sigma \;\; = e^{iu\hat{N}_\sigma} \, \frac{1}{\sqrt{\eta}} \, e^{i(\hat{\phi}^+_\sigma(u) + \hat{\phi}_\sigma(u))} \, \hat{U}_\sigma \tag{4}$$

in terms of the bosonization phases

$$\hat{\phi}_\sigma(u) = -i \sum_{m=1} \frac{1}{\sqrt{m}} \, e^{im(u + i\eta/2)} \, \hat{b}_{m\sigma}. \tag{5}$$

The quantity η in the non-normal ordered expression is a positive infinitesimal.

There is a nontrivial relation between the bosonization phase $\hat{\phi}_\sigma^+(u)+\hat{\phi}_\sigma(u)$ and the physical phase operators $\hat{\Phi}_\nu(u)$ and $\hat{\Theta}_\nu(u)$ given below. The Klein operator $\hat{U}_\sigma$ reduces the fermion number by one. An explicit construction of $\hat{k}_\sigma = -i\ln\hat{U}_\sigma$ is given in [32]. The zero modes in the physical phases $\hat{\Phi}_\nu$ and $\hat{\Theta}_\nu$ are related to $u\hat{N}_\sigma$ and $\hat{k}_\sigma$ by the transformation equation (1).

In the present case of the harmonic trap, the relation between the auxiliary variable u in the correlation functions and the spatial position z inside the trap turns out to be $u \to u_0(z) = \arcsin(z/L_F) - \pi/2$, where $2L_F$ is the quasi-classical extension of the Fermi sea.

The paper is organized as follows: Section 2 develops the phase theory for a one-component Fermi gas according to the above prescription. Section 3 extends the theory to two components. In Section 4, we calculate the scaling exponents contained in the one-particle correlation functions. The Appendix gives all details of their evaluation using the WKB representation developed in [34].

2. Phase theory for one component

We start with the simpler case of one fermionic component, e.g., spin polarized fermionic atoms in the case of ultracold atoms. The one-dimensional equivalent of s-wave scattering, i.e., a contact interaction is forbidden by the exclusion principle and the interaction in such a Fermi gas is usually weak. The model may, nevertheless, have potential application to ultracold Fermi gases in a harmonic trap provided Feshbach resonances enhance the interaction as recently found in the three-dimensional case [35].

Quasi one-dimensionality can be realized when the particle number N is less than the ratio of transverse to longitudinal trap frequency ω_t/ω_ℓ and the three-dimensional interaction energy per fermion is less than the transverse excitation energy in the highly elongated cylindrical trap [36].

However, it is not enough to reach degeneracy, $k_BT \ll \epsilon_F$: In order to observe zero temperature behavior, the much stronger condition $k_BT \ll \hbar\omega_\ell$ is also required.

The model considered in [17] is described by the bosonic excitation Hamiltonian

$$\tilde{H} = \frac{1}{2}\hbar\omega_\ell \sum_{m>0} m\left(\hat{d}_m\hat{d}_m^+ + \hat{d}_m^+\hat{d}_m\right) + \frac{1}{2}\sum_{m>0}\left[V_c(m) - V_b(m)\right]\sqrt{2m}\left[\hat{d}_{2m} + \hat{d}_{2m}^+\right] \tag{6}$$

$$+\frac{1}{2}\sum_{m>0} V_a(m)\, m\left(\hat{d}_m\hat{d}_m^+ + \hat{d}_m^+\hat{d}_m\right) - \frac{1}{2}\sum_{m>0}\left[V_c(m) - V_b(m)\right] m\left(\hat{d}_m^2 + \hat{d}_m^{+2}\right).$$

In the case of one component, one does not have to discriminate between $\hat{d}$ and $\hat{b}$ operators. We treat the one-particle operators from V_b and V_c on the same footing to avoid problems with spurious self-energies when V_b is retained.

The interaction terms in equation (6) follow uniquely by retaining those parts in the fermionic pair interaction operator

$$\hat{V} = \frac{1}{2} \sum_{mnpq} V(m,p;q,n)\,(\hat{c}_m^+ \hat{c}_q)(\hat{c}_p^+ \hat{c}_n), \tag{7}$$

which are expressible in terms of density fluctuation operators $\hat{\rho}(m) = \sum_p \hat{c}_{p+m}^+ \hat{c}_p$ (in the basis of harmonic oscillator states $\psi_n(z)$), i.e.,

$$V(m,p;q,n) \to V_a(|q-m|)\,\delta_{m-q,n-p} + V_b(|q-m|)\,\delta_{q-m,n-p} + V_c(|q-p|)\,\delta_{m+q,n+p}. \tag{8}$$

In [37], it is shown that the retained matrix elements are dominant in the one-dimensional many-fermion system in the harmonic trap. This is related to approximate momentum conservation during collisions in the trap.

The interaction matrix elements $V_a(m)$, $V_b(m)$, and $V_c(m)$ correspond to the Luttinger model coupling functions $g_4(p)$, $g_2(p)$, and $g_1(p)$, respectively. V_a and V_b describe forward scattering and V_c describes $2k_F$ (backward) scattering. The notion of backscattering in the one-branch-one-component system is to be interpreted as follows: The study of the matrix elements showed that the right (left) running part of a one-particle wave function interferes with the left (right) running part of the wave function of another fermion exchanging a momentum of about $2k_F$.

The drastic reduction of interaction matrix elements necessary to obtain equation (6) from the full Hamiltonian for a given one-dimensional (effective) pair interaction between fermions may affect physical details. However, the model will still provide us with the correct critical exponents in terms of the central coupling constant K. This has been explicitly shown in [33] for OBC. It must be pointed out that the model accounts for the basic interaction mechanisms in one dimension, namely forward and backward scattering. It may be more practical to consider K as a parameter taken from experiment, as is usually done in applications of the Luttinger model.

2.1. Phase operators

The second contribution on the r.h.s. of equation (6) represents a one-particle operator $\hat{V}_1$. This operator is neglected in the model with open boundary conditions as pointed out in [33]. The one-particle operator does not alter the critical behavior, but has quantitative effects on other properties. $\hat{V}_1$ is exactly taken into account in the present model. For more details, we refer to [17].

The Bogoliubov transformation to diagonalize the Hamiltonian in terms of operators $\hat{f}$ and $\hat{f}^+$ is given by $\hat{d}_m = \hat{f}_m \cosh\zeta_m - \hat{f}_m^+ \sinh\zeta_m + \eta_m \exp(-\zeta_m)$, where the inhomogeneous part $\eta_{2m} = -[V_c(m) - V_b(m)]\exp(-\zeta_{2m})/(2\epsilon_{2m}\sqrt{2m})$ originates from $\hat{V}_1$ and differs from zero only for even indices.

The central dimensionless coupling constants K_m, transformation parameters ξ_m, and the renormalized level spacings ϵ_m for the model, equation (6), are given by [17]

$$K_m = e^{-2\zeta_m} = \sqrt{\frac{\hbar\omega_\ell + V_a(m) - (V_b(m) - V_c(m))}{\hbar\omega_\ell + V_a(m) + (V_b(m) - V_c(m))}} \equiv \frac{K'_m}{2\pi}, \tag{9}$$

$$\epsilon_m \ = \sqrt{(\hbar\omega_\ell + V_a(m))^2 - (V_b(m) - V_c(m))^2},$$

respectively.

In the phase formulation, the dependence of K and ϵ on m must be suppressed. Note that $V_a(m) \to V_a$ implies that there are no interaction effects due to this matrix element in the one component theory and V_a should be omitted, i.e., we use $\epsilon = \sqrt{(\hbar\omega_\ell)^2 - (V_b - V_c)^2}$ and

$$K' = 2\pi K = 2\pi\, e^{-2\zeta} = 2\pi \sqrt{\frac{\hbar\omega_\ell - (V_b - V_c)}{\hbar\omega_\ell + (V_b - V_c)}}, \quad \epsilon = \hbar\omega_\ell \frac{2K}{K^2 + 1}, \tag{10}$$

where V_b is usually small in comparison to V_c, even in the marginally long-ranged case of the dipole-dipole interaction [37].

In some physical quantities, e.g., the one-particle correlation function, neglect of the dependence on m in K (which for large m must approach unity [4]) leads to divergences. We take care of this by writing

$$K_m = 1 + (K - 1)\, \exp(-mr), \tag{11}$$

with a small positive number $r \ll 1$, estimated as $r \approx R/L_F \propto 1/\sqrt{N}$ where R is the range of the interaction. This small number should be discriminated from the positive infinitesimal η appearing in equation (4). For $m < 1/r \gg 1$, there is practically no dependence on m.

We begin with rewriting the one-particle operator $\hat{V}_1$. This already leads to the structure of the main phase operator. Assuming the same exponential decay as in equation (11) also for interaction matrix elements, i.e., $V_c(m) - V_b(m) = (V_c - V_b)\, \exp(-mr)$, $\hat{V}_1$ can be expressed as

$$\hat{V}_1 = \frac{1}{4\pi}\,(V_c - V_b) \int_{-\pi}^{\pi} du \left[\frac{e^{-r+2iu}}{1 - e^{-r+2iu}} + \frac{e^{-r-2iu}}{1 - e^{-r-2iu}}\right] \partial_u \hat{\phi}_{odd}(u), \tag{12}$$

where

$$\hat{\phi}_{odd}(u) \equiv \frac{1}{2}\,(\hat{\phi}(u) + \hat{\phi}^+(u) - \hat{\phi}(-u) - \hat{\phi}^+(-u)) = \sum_{n=1}^{\infty} \sqrt{\frac{e^{-n\eta}}{n}}\ \sin(nu)\ \left(\hat{d}_n + \hat{d}_n^+\right). \tag{13}$$

The same phase operator appears in the particle density operator when it is decomposed into a slowly varying part and a part describing Friedel oscillations [34].

At the level of phase operators, the presence of $\hat{V}_1$ causes a c-number shift

$$b(u) = i\,\frac{K(V_c - V_b)}{4\epsilon}\,\ln\left(\frac{1-e^{-r+2iu}}{1-e^{-r-2iu}}\right) \equiv i\kappa_0\,\ln\left(\frac{1-e^{-r+2iu}}{1-e^{-r-2iu}}\right), \tag{14}$$

with $\kappa_0 = (K^2-1)/8$. The main phase operator $\hat{\Phi}(u)$ corresponds to that in [25]:

$$\hat{\Phi}(u) \equiv \hat{N}u + \hat{\phi}_{odd}(u) + b(u) = \hat{N}u + \sum_{m=1}^{\infty}\sqrt{\frac{K_m\,e^{-m\eta}}{m}}\,\sin(mu)\,(\hat{f}_m + \hat{f}_m^+). \tag{15}$$

Note, that in [25] rescaled phases are used and the function $b(u)$ is absent due to the neglect of one-particle operators. The dual phase operator is

$$\hat{\Theta}(u) \equiv \frac{i}{2\pi}\,\sum_{n=1}^{\infty}\sqrt{\frac{e^{-n\eta}}{n}}\,\cos(nu)\,\left(\hat{d}_n - \hat{d}_n^+\right) = \frac{i}{2\pi}\,\sum_{n=1}^{\infty}\sqrt{\frac{e^{-n\eta}}{nK_n}}\,\cos(nu)\,\left(\hat{f}_n - \hat{f}_n^+\right). \tag{16}$$

We prefer not to combine the zero mode operator $\hat{k}$ with the dual phase field $\hat{\Theta}$. Instead, we retain the unitary operator $\hat{U} = \exp(i\hat{k})$ in order to avoid the mathematical problems pointed out in [38]. The relevant commutator is $[\exp(i\hat{k}), \hat{N}] = \exp(i\hat{k})$. The phase operator corresponding to the momentum density is

$$\hat{\Pi}(u) \equiv \partial_u\hat{\Theta}(u) = -\frac{i}{2\pi}\,\sum_{n=1}^{\infty}\sqrt{n\,e^{-n\eta}}\,\sin(nu)\,(\hat{d}_n - \hat{d}_n^+) \tag{17}$$

$$= -\frac{i}{2\pi}\,\sum_{n=1}^{\infty}\sqrt{\frac{n\,e^{-n\eta}}{K_n}}\,\sin(nu)\,(\hat{f}_n - \hat{f}_n^+).$$

Using $\sum_{m=1}^{\infty}\left(e^{im(u+i\eta/2)} + e^{-im(u-i\eta/2)}\right) \to 2\pi\,\delta_{2\pi}(u) - 1$ leads to the commutator

$$\left[\hat{\Phi}(u), \hat{\Pi}(v)\right] = \frac{i}{2}\,(\delta_{2\pi}(u-v) - \delta_{2\pi}(u+v))\,, \tag{18}$$

which is also implicit in the work [25]. This is not a canonical commutator. Canonical commutation relations cannot hold for operators with fixed parity.

2.2. Phase Hamiltonian

Inserting equation (15) and equation (17) into

$$\hat{H} = \frac{\epsilon}{2}\,\int_{-\pi}^{\pi}du\left[K'\,\hat{\Pi}^2(u) + \frac{1}{K'}\left(\partial_u\hat{\Phi}(u)\right)^2\right] = \int_{-\pi}^{\pi}du\,\hat{\mathcal{H}}, \tag{19}$$

and neglecting an irrelevant contribution from $b^2(u)$, reproduces equation (6) in addition to the zero mode contribution $E_0(\hat{N})$

$$E_0(\hat{N}) = \frac{\epsilon}{2K}\,\hat{N}^2 = \frac{\hbar\omega_\ell}{K^2+1}\,\hat{N}^2. \tag{20}$$

In the latter relation, corrections $O(\hat{N})$ are neglected. Remarkably, canonical equations still hold in spite of equation (18): Consider the Heisenberg phase field $\hat{\Phi}(u,t)$ (we set $\hbar = 1$)

$$i\partial_t\hat{\Phi}(u,t) = \left[\hat{\Phi}(u,t), \hat{H}(t)\right] = i\epsilon\,K'\,\hat{\Pi}(u,t) \equiv i\frac{\partial\hat{\mathcal{H}}}{\partial\hat{\Pi}(u,t)}. \tag{21}$$

Thus $\partial_t\hat{\Phi}(u,t) = \epsilon\,K'\,\hat{\Pi}(u,t)$ remains correct.

2.3. Bosonization

A simple expression of the individual operators $\hat{\phi}$ and $\hat{\phi}^+$ in equation (4) in terms of the new phase operators $\hat{\Phi}$ and $\hat{\Theta}$ is not available. However, their sum which appears in the non-normal ordered form of equation (4), is represented by

$$\hat{\phi}(u) + \hat{\phi}^+(u) = \hat{\phi}_{odd}(u) - 2\pi\,\hat{\Theta}(u) \equiv \hat{\Phi}(u) - b(u) - \hat{N}u - 2\pi\,\hat{\Theta}(u). \tag{22}$$

This can be utilized to calculate the auxiliary correlation function

$$\langle\hat{\psi}_a^+(u,t)\psi_a(v)\rangle = e^{-i(N-1)(u-v)+i\mu_N t}\,\frac{1}{\eta}\langle e^{-i\hat{\phi}^+(u,t)-i\hat{\phi}(u,t)}e^{i\hat{\phi}^+(v)+i\hat{\phi}(v)}\rangle, \tag{23}$$

with the chemical potential

$$\mu_N \equiv E_0(N) - E_0(N-1) = \frac{2\hbar\omega_\ell}{K^2+1}\,\left(N - \frac{1}{2}\right). \tag{24}$$

Analogously

$$\langle\hat{\psi}_a(u,t)\hat{\psi}_a^+(v)\rangle = e^{iN(u-v)-i\mu_{N+1}t}\,\frac{1}{\eta}\langle e^{i\hat{\phi}^+(u,t)+i\hat{\phi}(u,t)}e^{-i\hat{\phi}^+(v)-i\hat{\phi}(v)}\rangle. \tag{25}$$

Derivation and evaluation of these expressions is described in the Appendix.

3. Phase Theory for two components

In terms of mass and composition operators, i.e., after the transformation equation (1), the excitation Hamiltonian for forward scattering is given by

$$\begin{aligned}\tilde{H}_{for} \equiv\; & \frac{1}{2}\sum_{m\nu} m[\hbar\omega_\ell + V_{a\|}(m) + \nu V_{a\perp}(m)]\left\{\hat{d}_{m\nu}^+\hat{d}_{m\nu} + \hat{d}_{m\nu}\hat{d}_{m\nu}^+\right\} \\ & + \frac{1}{2}\sum_{m,\nu} m[V_{b\|}(m) + \nu V_{b\perp}(m)]\left\{\hat{d}_{m\nu}^{+2} + \hat{d}_{m\nu}^2\right\}.\end{aligned} \tag{26}$$

It must be supplemented by backward scattering and one-particle contributions. Noting that the one-particle operator $\frac{1}{2}\sum_{m>0,\sigma}\sqrt{2m}\,(V_{c\|}(m)-V_{b\|}(m))\,[\hat{b}_{2m\sigma}+\hat{b}^{+}_{2m\sigma}]$, which originates from rearranging the two-particle backscattering operator and from a spurious self-energy in $\hat{V}_b$, is transformed by means of $\sum_\sigma \sigma^{\frac{1-\nu}{2}} \equiv 1+\nu = 2\delta_{\nu,1}$ and

$$\sum_\sigma [\hat{b}_{m\sigma}+\hat{b}^{+}_{m\sigma}] = \frac{1}{\sqrt{2}}\sum_\nu \left(\sum_\sigma \sigma^{\frac{1-\nu}{2}}\right)\left(\hat{d}_{m\nu}+\hat{d}^{+}_{m\nu}\right) = \sqrt{2}\left(\hat{d}_{m1}+\hat{d}^{+}_{m1}\right), \tag{27}$$

the final form of the additional operator is

$$\hat{V}_{add} = -\frac{1}{2}\sum_{m>0,\nu} mV_{c\|}(m)\left\{\hat{d}^{+2}_{m\nu}+\hat{d}^{2}_{m\nu}\right\} + \sum_{m>0}\sqrt{m}[V_{c\|}(m)-V_{b\|}(m)]\left(\hat{d}^{+}_{2m1}+\hat{d}_{2m1}\right). \tag{28}$$

So far, we did not mention $V_{c\perp}$, i.e., backscattering between the two components, but we note that $V_{c\perp}$ does not generate a one-particle potential.

In omitting the dependence on m, $V_{a\|}$ drops out again. Then the renormalized level spacings and central coupling constants are given by

$$\epsilon_\nu = \sqrt{(\hbar\omega_\ell+\nu V_{a\perp})^2-[V_{b\|}+\nu V_{b\perp}-V_{c\|}]^2}, \tag{29}$$

and

$$K_\nu = \sqrt{\frac{(\hbar\omega_\ell+\nu V_{a\perp})-[V_{b\|}+\nu V_{b\perp}-V_{c\|}]}{(\hbar\omega_\ell+\nu V_{a\perp})+[V_{b\|}+\nu V_{b\perp}-V_{c\|}]}} \equiv \frac{K'_\nu}{2\pi}, \tag{30}$$

respectively. The total one-particle operator from V_b and V_c can be rewritten as

$$\hat{V}_1 = \frac{1}{4\pi\sqrt{2}}\,(V_{c\|}-V_{b\|})\,\sum_\nu(1+\nu)\int_{-\pi}^{\pi}du\,\left[\frac{e^{-r+2iu}}{1-e^{-r+2iu}}+\frac{e^{-r-2iu}}{1-e^{-r-2iu}}\right]\partial_u\hat{\phi}_{\nu,odd}(u), \tag{31}$$

with

$$\hat{\phi}_{\nu,odd}(u) \equiv \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\,e^{-n\eta/2}\,\sin(nu)\left(\hat{d}_{n\nu}+\hat{d}^{+}_{n\nu}\right). \tag{32}$$

The corresponding phase shifts are the c-number functions (real and odd)

$$b_\nu(u) = i\,\frac{K'_\nu(V_{c\|}-V_{b\|})}{8\pi\epsilon_\nu\sqrt{2}}(1+\nu)\,\ln\left(\frac{1-e^{-r+2iu}}{1-e^{-r-2iu}}\right) \equiv i\kappa_1\,\delta_{\nu,1}\ln\left(\frac{1-e^{-r+2iu}}{1-e^{-r-2iu}}\right), \tag{33}$$

with $\kappa_1 \equiv (K_1^2-1)/8$. Because of $b_{-1}=0$ only the mass part is affected by the one-particle operator. The phase operators are

$$\hat{\Phi}_\nu(u) \equiv \hat{\phi}_{\nu,odd}(u)+b_\nu(u)+u\hat{N}_\nu = \sum_{n=1}^{\infty}\sqrt{\frac{K_{n\nu}}{n}}\,e^{-n\eta/2}\,\sin(nu)\,(\hat{f}_{n\nu}+\hat{f}^{+}_{n\nu})+u\hat{N}_\nu, \tag{34}$$

with $\hat{N}_\nu = \sum_\sigma \sigma^{\frac{1-\nu}{2}}\, \hat{N}_\sigma/\sqrt{2}$ and the dual phase operator

$$\hat{\Theta}_\nu(u) \equiv \frac{i}{2\pi} \sum_{n=1}^{\infty} \sqrt{\frac{e^{-n\eta}}{n}}\, \cos(nu)\, \left(\hat{d}_{n\nu} - \hat{d}^+_{n\nu}\right) = \frac{i}{2\pi} \sum_{n=1}^{\infty} \sqrt{\frac{e^{-n\eta}}{nK_{n\nu}}}\, \cos(nu)\, (\hat{f}_{n\nu} - \hat{f}^+_{n\nu}). \quad (35)$$

The dual phase fields $\hat{\Theta}_\nu$ generate associated momentum densities via

$$\hat{\Pi}_\nu(u) \equiv \frac{\partial}{\partial u}\, \hat{\Theta}_\nu(u) = -\frac{i}{2\pi} \sum_{n=1}^{\infty} \sqrt{ne^{-n\eta}}\, \sin(nu)\, \left(\hat{d}_{n\nu} - \hat{d}^+_{n\nu}\right). \quad (36)$$

Finally, the phase Hamiltonian is

$$\hat{H} = \sum_\nu \frac{\epsilon_\nu}{2} \int_{-\pi}^{\pi} du \left\{ K'_\nu \hat{\Pi}^2_\nu(u) + \frac{1}{K'_\nu} \left(\partial_u \hat{\Phi}_\nu(u)\right)^2 \right\}. \quad (37)$$

The zero mode in $\hat{\Phi}_\nu$ gives the ground state energy operator

$$\hat{E}_{0\nu} = \frac{\epsilon_\nu}{4K_\nu} \left(\delta_{\nu,1} \sum_{\sigma\sigma'} \hat{N}_\sigma\, \hat{N}_{\sigma'} + \delta_{\nu,-1} \left(\sum_\sigma \sigma \hat{N}_\sigma \right)^2 \right). \quad (38)$$

Again, the equations of motion for the Heisenberg fields $\hat{\Phi}_\nu(u,t)$ are $\partial_t \hat{\Phi}_\nu(u,t) = \epsilon_\nu\, K'_\nu\, \hat{\Pi}_\nu(u,t)$.

Bosonization according to equation (4) is provided by

$$\hat{\phi}_\sigma(u) + \hat{\phi}^+_\sigma(u) = \sum_\nu \frac{\sigma^{\frac{1-\nu}{2}}}{\sqrt{2}} \left(\hat{\Phi}_\nu(u) - b_\nu(u) - u\hat{N}_\nu - 2\pi\, \hat{\Theta}_\nu(u)\right). \quad (39)$$

Backscattering between different components is described by the operator $\hat{V}_{c\perp}$. The composition part $\hat{H}_{\nu=-1}$ acquires the form of a sine-Gordon Hamiltonian on the unit circle. Considering the results for the usual sine-Gordon system, one can speculate that a sufficiently repulsive interaction renders $\hat{V}_{c\perp}$ irrelevant at low energies. Restricting the treatment to long range interactions as discussed in [33] avoids this problem.

4. Critical exponents

This section compiles the results for critical exponents extracted from the one-particle correlation functions which are calculated in the Appendix. Subsections are devoted to the static two-point correlation function, the local dynamic correlation function, and to the local spectral density of states, respectively.

4.1. Static two-point correlation function

We consider the zero temperature static correlation function

$$C(z_1, z_2) \equiv C(z_1, t = 0; z_2) \equiv \langle \hat{\psi}^+(z_1)\, \hat{\psi}(z_2) \rangle_0. \tag{40}$$

We have to evaluate equation (A.11) using the explicit result (A.17) for $t = 0$. To this order, we introduce the abbreviations $Z_\nu \equiv (1 - z_\nu^2/L_F^2)^{1/2}$; $\quad \nu = 1, 2$, $L_F = \ell\sqrt{2N-1}$ (ℓ is the oscillator length) to find for $L_F - |z_\nu| > L_F r^2$ and $z_1 \neq z_2$ (the case $z_1 = z_2$ was considered in [34])

$$|\mathcal{D}(\pm 2u_0(z_\nu) + ir)|^2 \to \frac{1}{4Z_\nu^2},\ |\mathcal{D}(u_0(z_1) \pm u_0(z_2) + ir)|^2 \to \frac{1}{2\,(1 \pm Z_1 Z_2 - z_1 z_2/L_F^2)}. \tag{41}$$

This gives

$$C(z_1, z_2) = \frac{r^{2\gamma_0} 2^{\alpha_0 + \frac{1}{2} - \gamma_0}}{2\pi L_F} \left[\frac{\sin(k_F(\tilde{z}_1 - \tilde{z}_2) + A_-)}{(1 - Z_1 Z_2 - z_1 z_2/L_F^2)^{\gamma_0 + \frac{1}{2}} (1 + Z_1 Z_2 - z_1 z_2/L_F^2)^{\alpha_0}} \right. \tag{42}$$

$$\left. + (-1)^{N-1} \frac{\cos(k_F(\tilde{z}_1 + \tilde{z}_2) + A_+)}{(1 + Z_1 Z_2 - z_1 z_2/L_F^2)^{\gamma_0 + \frac{1}{2}} (1 - Z_1 Z_2 - z_1 z_2/L_F^2)^{\alpha_0}} \right] (Z_1 Z_2)^{\alpha_0 - \frac{1}{2}},$$

with the (irrelevant) phase shift $A_\pm \equiv \frac{1}{2}\,(\arcsin(z_1/L_F) \pm \arcsin(z_2/L_F)) - b(u_0(z_1)) \mp b(u_0(z_2))$ and notations from the Appendix, e.g., $\tilde{z}_\nu = \tilde{z}(z_\nu) = z_\nu Z(z_\nu)/2 + (L_F/2)\arcsin(z_\nu/L_F)$. For $z_1 \equiv z > \ell$ and $z_2 = 0$, and well inside the trap ($|z| \ll L_F$), we find

$$C_{\text{center}}(z) = \frac{r^{2\gamma_0}}{\pi L_F} \frac{\sin k_F z}{(z/L_F)^{2\gamma_0 + 1}} + (-1)^{N-1} \frac{r^{2\gamma_0}}{2\pi L_F} 2^{2\alpha_0 - 2\gamma_0} \frac{\cos k_F z}{(z/L_F)^{2\alpha_0}}. \tag{43}$$

Because of $|z|/L_F \ll 1$ and $2\gamma_0 + 1 = 2\alpha_0 + K$, the second term is not important. Thus

$$C_{\text{center}}(z) = \frac{r^{2\gamma_0}}{\pi L_F} \frac{\sin k_F z}{(z/L_F)^{2\gamma_0 + 1}}, \tag{44}$$

gives the decay exponent (twice the bulk scaling dimension of $\hat{\psi}$)

$$\alpha_C = 2\gamma_0 + 1 = \frac{1}{2}\left[K + \frac{1}{K}\right] \equiv 2\Delta. \tag{45}$$

This result was obtained numerically for the present model in the second paper of [17]. It coincides not unexpectedly with the OBC result [24, 25, 33] in the bulk limit and also with that for the homogeneous Luttinger model.

We can extract from equation (42) the boundary exponent $\Delta_\perp$. To this order, the thermodynamic limit must be performed explicitly: As in [39] this is done by making

the trap shallower and shallower: $\omega_\ell \propto 1/N \to 0$. Furthermore, the coupling constant K and the Fermi wave number k_F must be held fixed. For a specific interaction model, this limit was already performed in [34]. Furthermore, besides η also $r \propto 1/\sqrt{N}$ vanishes and one should consider renormalized fields as described in [40].

Consider then the boundary situation $z_2 = L_F(1-\eta)$ while $0 < z_1 \ll L_F$, such that $z_2 \to L_F$ that is $Z_2 \to 0$ in the denominator of $C(z_1, z_2)$. The distance $z_2 - z_1$ then is large as required for $\Delta_\perp$. Equation (42) gives

$$C(z_1, L_F(1-\eta)) \propto \frac{Z_1^{\alpha_0 - 1/2}}{(1 - z_1/L_F)^{\alpha_0+\gamma_0+\frac{1}{2}}} \, (\sqrt{\eta})^{\alpha_0-1/2} \propto \frac{(z_2 - z_1)^{(\alpha_0-1/2)/2}}{(z_2 - z_1)^{\alpha_0+\gamma_0+\frac{1}{2}}}. \tag{46}$$

Hence

$$2\Delta_\perp = \frac{1}{2}\,\alpha_0 + \gamma_0 + \frac{3}{4} = \frac{1}{8}\,\left[\frac{3}{K} + K + 2\right]. \tag{47}$$

This boundary exponent is different from the corresponding result for OBC in [24, 25].

4.2. Local dynamic correlation function

Following [25, 33], we consider only the part varying slowly in space, i.e., we neglect the rapidly oscillating Friedel part given by the last two terms in equation (A.11). We then obtain

$$C^{NF}(z,t;z) = \frac{1}{\pi L_F Z(z)} \langle \hat{\psi}_a^+(u_0(z),t)\hat{\psi}_a(u_0(z))\rangle_0. \tag{48}$$

According to equation (A.17),

$$\langle \hat{\psi}_a^+(u_0,t)\hat{\psi}_a(u_0)\rangle_0 = e^{i\mu_N t}\, r^{2\gamma_0}\, \left[\mathcal{D}(-\epsilon t + ir)^{2\gamma_0}\, \mathcal{D}(-\epsilon t + i\eta)\right] \tag{49}$$

$$\times \left\{ \frac{\mathcal{D}(2u_0 - \epsilon t + ir)\, \mathcal{D}(-2u_0 - \epsilon t + ir)}{|\mathcal{D}(2u_0 + ir)|^2} \right\}^{\alpha_0}$$

holds with $u_0 = \arcsin(z/L_F) - \pi/2$. The renormalized level spacing ϵ vanishes in the thermodynamic limit, but the chemical potential becomes a constant independent of particle number: $\mu_N = \epsilon\,(N - 1/2)/K \to 2\hbar/(K^2+1)\,(\omega_\ell N) = const.$. Vanishing η and r imply

$$\mathcal{D}(-\epsilon t) = -\frac{ie^{i\epsilon t/2}}{2\sin(\epsilon t/2)}, \quad \mathcal{D}(2u_0 - \epsilon t)\mathcal{D}(-2u_0 - \epsilon t) = \frac{e^{i\epsilon t}}{4\,[\cos^2(\epsilon t/2) - z^2/L_F^2]}. \tag{50}$$

In order to collect all possible powers of t, we specify z to be so near the classical boundaries that $\cos^2(\epsilon t/2) - z^2/L_F^2 \propto t^2$ holds. Then we obtain:

$$C^{NF} \propto \mathcal{D}(-\epsilon t)^{2\gamma_0+2\alpha_0+1} \propto \frac{1}{t^{2\Delta_\parallel}}, \tag{51}$$

with the boundary exponent

$$2\Delta_\parallel = 2\gamma_0 + 2\alpha_0 + 1 = \frac{1}{K}. \tag{52}$$

This agrees with the one-component results in [24, 25, 33] for OBC.

However, for z away from the boundaries, only $\mathcal{D}(-\epsilon t)^{2\gamma_0+1}$ contributes and the bulk scaling exponent is reobtained

$$2\gamma_0 + 1 = \frac{1}{2}\left[K + \frac{1}{K}\right] \equiv 2\Delta. \tag{53}$$

Note that the scaling relation $2\Delta_\perp = \Delta + \Delta_\parallel$ [41] is not fulfilled with respect to the variables z and t. However, boundary conformal field theory [42] applies to the auxiliary model which uses the variables $\epsilon\tau$ ($it \equiv \tau$) and u: The infinite strip $w \equiv \epsilon\tau - iu$ with $-\pi \leq u \leq 0$ can be mapped conformally onto the complete complex plane and the Euclidean Lagrange density derived from equation (19) is locally rotation invariant. Consequently, the scaling relation holds irrespective of the boundary condition. This can be checked by direct calculation: If we calculate $\langle\hat{\psi}_a^+(u \approx -\pi/2)\hat{\psi}_a(v \approx 0, \pi)\rangle_0$ instead of the physical correlation function $C(z_1 \approx 0, z_2 \approx L_F, -L_F)$ in order to get the boundary exponent $\Delta_\perp^{(a)}$ for the auxiliary fields we find

$$\langle\hat{\psi}_a^+(u)\,\hat{\psi}_a(v)\rangle_0 \propto \frac{1}{\left|\sin(\frac{u-v}{2})\right|^{2\Delta_\perp^{(a)}}}, \tag{54}$$

with the new value

$$2\Delta_\perp^{(a)} = \frac{1}{4}\left[K + \frac{3}{K}\right]. \tag{55}$$

The critical exponents Δ and $\Delta_\parallel$ remain unchanged so that the scaling relation is fulfilled.

The subsequent transformation to the physical plane $z_c \equiv \bar{z} + i\epsilon\tau$ ($\bar{z} \equiv z/L_F$) which must ensure $\bar{z} = \cos u$ is, however, not conformal. The breakdown of local conformal invariance with respect to the scaled coordinates $\bar{z}$ and $\epsilon\tau$ near the boundaries becomes obvious by inspection of the corresponding Euclidean Lagrange density

$$\hat{\mathcal{L}}_E = \frac{\epsilon}{2K'}\left\{\left(\partial_{\epsilon\tau}\hat{\Phi}\right)^2 + Z^2\left(\partial_{\bar{z}}\hat{\Phi}\right)^2\right\}, \tag{56}$$

with $Z^2 = 1 - \bar{z}^2$.

4.3. Local spectral density

We must evaluate the anti-commutator omitting the Friedel part

$$A(t,z) \equiv \left\langle \left[\hat{\psi}^+(z,0), \hat{\psi}(z,t)\right]_+ \right\rangle_0^{NF}. \tag{57}$$

Using equation (A.17) and equation (A.19) of the Appendix, we find

$$A(t,z) = \left(e^{-i\mu_{N+1}t}\mathcal{D}(-\epsilon t + ir)^{2\gamma_0}\mathcal{D}(-\epsilon t + i\eta)\left\{\frac{\mathcal{D}(-2u_0 - \epsilon t + ir)\mathcal{D}(2u_0 - \epsilon t + ir)}{|\mathcal{D}(2u_0)|^2}\right\}^{\alpha_0} \right. \tag{58}$$

$$\left. + e^{-i\mu_N t}\,\mathcal{D}(\epsilon t + ir)^{2\gamma_0}\,\mathcal{D}(\epsilon t + i\eta)\left\{\frac{\mathcal{D}(-2u_0 + \epsilon t + ir)\mathcal{D}(2u_0 + \epsilon t + ir)}{|\mathcal{D}(2u_0)|^2}\right\}^{\alpha_0} \right) \frac{r^{2\gamma_0}}{\pi L_F Z}.$$

In the thermodynamic limit, $r \to 0$, $\mu_N \to \mu_{N+1} \to \mu$, and measuring the frequency ω from the chemical potential, $\omega - \mu \to \omega$, we obtain

$$N(\omega,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\, A(t,z) \propto \frac{Z^{2\alpha_0-1}(z)2^{-2\gamma_0-1}}{\pi^2 L_F} \tag{59}$$

$$\times \int_0^{\infty} dt \cos(\omega t)\left\{\frac{e^{-i(\gamma_0+1/2)\pi + i(\gamma_0+1/2+\alpha_0)\epsilon t}}{\sin^{2\gamma_0+1}(t\epsilon/2)\,[\cos^2(t\epsilon/2) - z^2/L_F^2]^{\alpha_0}} + c.c.\right\}.$$

We further proceed as in [25]: The regularization $\cos(\omega t) \to [\cos(\omega t) - 1]$ is adopted and the integral $\int_0^\infty dt[\cos(t) - 1]/t^k = \pi/(2\Gamma(k)\cos(\pi k/2)), \quad 1 < k < 3$ is used. Well inside the trap, we find

$$N(\omega, |z| \ll L_F) \propto \frac{1}{\pi L_F}\frac{1}{\epsilon\Gamma(2\gamma_0+1)}\left(\frac{\omega}{\epsilon}\right)^{2\gamma_0}. \tag{60}$$

The bulk exponent of the local spectral density thus is

$$\alpha_{\rm bulk} = 2\gamma_0 = \frac{1}{2}\left[K + \frac{1}{K} - 2\right], \quad 0 < \gamma_0 < 1, \tag{61}$$

and agrees with the corresponding result in [25, 33] for OBC.

The boundary exponent is obtained as follows: Near a boundary, $|z| \approx L_F$, we obtain for $4Z^2 \ll \epsilon^2 t^2 \ll 1$, when $[\cos^2(t\epsilon/2) - z^2/L_F^2]^{\alpha_0} \to e^{i\pi\alpha_0}2^{-2\alpha_0}(t\epsilon)^{2\alpha_0}$,

$$N(\omega, |z| \approx L_F) \propto \frac{1}{\pi L_F}\frac{2^{2\alpha_0}\, Z^{2\alpha_0-1}(z)}{\epsilon\,\Gamma(2\gamma_0+2\alpha_0+1)}\left(\frac{\omega}{\epsilon}\right)^{2\gamma_0+2\alpha_0}. \tag{62}$$

The boundary critical exponent thus is

$$\alpha_{\rm boundary} = 2\gamma_0 + 2\alpha_0 = \frac{1}{K} - 1, \quad 0 < \alpha_0 + \gamma_0 < 1. \tag{63}$$

Again, this value agrees with the one-component version extracted from [24, 25, 33] for OBC.

4.4. Two components

It is straightforward to extend these results to the case of two components: The correlation function factorizes in the mass and composition representation. However, the corresponding exponents in equation (A.12) have half the previous value due to the transformation equation (1). This leads to

$$2\Delta = \frac{1}{4}\left[K_1 + K_{-1} + \frac{1}{K_1} + \frac{1}{K_{-1}}\right], \quad 2\Delta_{\parallel} = \frac{1}{2}\left[\frac{1}{K_1} + \frac{1}{K_{-1}}\right], \tag{64}$$

$$2\Delta_{\perp} = \frac{1}{16}\left[K_1 + K_{-1} + \frac{3}{K_1} + \frac{3}{K_{-1}} + 4\right].$$

For the spectral functions, we assume equal particle numbers of the two components. Then the spectral function calculated above (and amended by the correction just described) refers to each component giving

$$\alpha_{\text{bulk}} = \frac{1}{4}\left[K_1 + K_{-1} + \frac{1}{K_1} + \frac{1}{K_{-1}} - 4\right], \quad \alpha_{\text{boundary}} = \frac{1}{2}\left[\frac{1}{K_1} + \frac{1}{K_{-1}} - 2\right]. \tag{65}$$

These values agree fully with those found in [24, 25, 33], except for the value of $\Delta_{\perp}$. Again, this results from the violation of local Lorentzian invariance with respect to spatial position and time in the present model.

5. Summary

We gave a consistent phase operator formulation for a model of interacting one-dimensional fermions in a harmonic trap. The model is the analogue of the Tomonaga-Luttinger model with open boundary conditions (OBC). Suitable phase operator were identified by expressing the one-particle operators as linear odd combinations of the basic bosonization operators in agreement with earlier results on the decomposition of the particle density operator. The total Hamiltonian became a quadratic form in the phase operator $\hat{\Phi}$ and the operator $\hat{\Pi}$ corresponding to the momentum density in spite of the presence of one-particle operators and the harmonic trap potential in the original fermionic Hamiltonian. These results were extended to the case of two components. The phase formulation bears many similarities to that for OBC. However, distinctive differences are also present. The spatial position is non-linearly related to the variable in the phase fields. This spoils local Lorentzian invariance usually satisfied by related phase theories. In addition, form factors appear which take account of the harmonic trap topology. Exact results for the one-particle correlation functions were derived and used to extract bulk and boundary critical exponents including those for the local spectral density. The values of the critical exponents coincide with those for the OBC except for the boundary scaling dimension of $\hat{\psi}$.

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Appendix: Calculation of one-particle correlation functions

5.1. Bosonization and auxiliary correlation function

We use the representation of auxiliary Fermi operators [32]

$$\hat{\psi}_a^+(u) = \hat{O}^+(u)\, e^{-i\hat{\phi}^+(u)}\, e^{-i\hat{\phi}(u)} = \frac{1}{\sqrt{\eta}}\, \hat{U}^+\, e^{-i\left(\hat{\phi}_{odd}(u)+\hat{N}u-2\pi\hat{\Theta}(u)\right)}. \tag{A.1}$$

The Klein operator $\hat{O}(u) = \exp(i\hat{N}u)\,\hat{U}$ acts on the fermion number operator $\hat{N}$. $\hat{U}$ commutes with all bosonic operators $\left[\hat{U},\hat{\phi}\right]_- = 0 = \left[\hat{U},\hat{\phi}^+\right]_-$, etc. and changes the fermion number according to $f(\hat{N})\,\hat{U}^+ = \hat{U}^+ f(\hat{N}+\hat{1})$. The contribution $\hat{N}u$ to the full phase operator in equation (15) leads via equation (19) to the ground state energy equation (20) and consequently to the chemical potential equation (24). We consider the one-particle correlation function of auxiliary Fermi operators

$$C_a(u,t;v) \equiv \ {}_N\langle\hat{\psi}_a^+(u,t)\hat{\psi}_a(v,0)\rangle_N, \tag{A.2}$$

evaluated with respect to an N-fermion state. The bosonized form is

$$C_a(u,t;v) = \frac{1}{\eta}\ {}_N\langle\hat{O}^+(u,t)\, e^{-i(\hat{\phi}^+(u,t)+\hat{\phi}(u,t))}\, e^{i(\hat{\phi}^+(v)+\hat{\phi}(v))}\hat{O}(v)\rangle_N. \tag{A.3}$$

The time evolution operator is governed by the separated Hamiltonian

$$\hat{H} = \tilde{H} + E_0(\hat{N}) = \sum_m m\epsilon_m\, \hat{f}_m^+\hat{f}_m + E_0(\hat{N}), \tag{A.4}$$

made up of the ground state energy of the N-fermion system plus the Hamiltonian of collective excitations. We thus get

$$C_a(u,t;v) = e^{-i(N-1)(u-v)}\,\frac{1}{\eta}\ {}_N\langle e^{-i\left(\hat{\phi}_{odd}(u,t)-2\pi\hat{\Theta}(u,t)\right)}\,\hat{U}^+(t)\,\hat{U}\, e^{i\left(\hat{\phi}_{odd}(v)-2\pi\hat{\Theta}(v)\right)}\rangle_N. \tag{A.5}$$

In order to determine $\hat{U}^+(t)$, we set $\hbar = 1$ and $f(\hat{N},t) \equiv \exp\left[iE_0(\hat{N})t\right]$ and find

$$\begin{aligned}\hat{U}^+(t) &= f(\hat{N},t)\,\hat{U}^+\, f^*(\hat{N},t) = \hat{U}^+\, f(\hat{N}+\hat{1},t)\, f^*(\hat{N},t) \\ &= \hat{U}^+\, \exp\left[i\left(E_0(\hat{N}+\hat{1}) - E_0(\hat{N})\right)t\right] = \hat{U}^+ \exp\left[i\mu_{\hat{N}+\hat{1}}t\right] = e^{i\mu_{\hat{N}}t}\,\hat{U}^+.\end{aligned} \tag{A.6}$$

Thus the time dependence of $\hat{U}^+$ is provided by the chemical potential: $\hat{U}^+(t) = \exp(i\mu_{\hat{N}}t)\,\hat{U}^+$. The final result for C_a is:

$$C_a(u,t;v) = \frac{1}{\eta}\, e^{-i(N-1)(u-v)+\frac{i}{\hbar}\mu_N t}\, \langle e^{-i\left(\hat{\phi}_{odd}(u,t)-2\pi\hat{\Theta}(u,t)\right)}\, e^{i\left(\hat{\phi}_{odd}(v)-2\pi\hat{\Theta}(v)\right)}\rangle. \tag{A.7}$$

The remaining expectation value is purely bosonic and can also be thermal.

5.2. Calculation of one-particle correlation function

The next step is to actually calculate the physical one-particle correlation function

$$C(z_1,t;z_2) \equiv \sum_{m,n} \psi_m(z_1)\psi_n(z_2) \int_{-\pi}^{\pi} \frac{dudv}{4\pi^2}\, e^{imu-inv} \langle \hat{\psi}_a^+(u,t)\hat{\psi}_a(v)\rangle \tag{A.8}$$

$$= \sum_{m,n} \psi_m(z_1)\psi_n(z_2) \int_{-\pi}^{\pi} \frac{dudv}{4\pi^2}\, e^{imu-inv} e^{-i(N-1)(u-v)+i\mu_N t} \langle e^{-i\hat{\phi}^+(u,t)}\, e^{-i\hat{\phi}(u,t)} e^{i\hat{\phi}^+(v)}\, e^{i\hat{\phi}(v)}\rangle.$$

This expression contains sums over harmonic oscillator wave functions $\psi_m(z)$. In [34], a WKB method was proposed to deal with such sums: For large N, the (singular) expansion

$$\sum_{m=1}^{\infty} \psi_m(z_1)\, e^{imu} \to \left(\frac{2\pi^2\alpha^2}{NZ^2(z_1)}\right)^{1/4} e^{i(N-1)u} \tag{A.9}$$

$$\times \left\{ e^{ik_F\tilde{z}(z_1)-i\pi(N-1)/2}\delta_{2\pi}(u+u_0(z_1)) + e^{-ik_F\tilde{z}(z_1)+i\pi(N-1)/2}\delta_{2\pi}(u-u_0(z_1)) \right\}$$

was found. Here, the following abbreviations are used

$$u_0(z) \equiv \arcsin\left(\frac{z}{L_F}\right) - \frac{\pi}{2}, \quad \tilde{z} = \frac{1}{2}zZ(z) + \frac{1}{2}L_F \arcsin\frac{z}{L_F}. \tag{A.10}$$

The spatial coordinates z_1 and z_2 are thus restricted to the classical region $(-L_F, L_F)$. The correlation function takes on the much simpler form

$$C(z_1,t;z_2) = \frac{\alpha\pi}{4\pi^2}\left(\frac{4}{N^2Z^2(z_1)Z^2(z_2)}\right)^{1/4} e^{i\mu_N t} \tag{A.11}$$

$$\times \Big\{ e^{ik_F(\tilde{z}(z_1)-\tilde{z}(z_2))} C_a(-u_0(z_1),t;-u_0(z_2)) + e^{-ik_F(\tilde{z}(z_1)-\tilde{z}(z_2))} C_a(u_0(z_1),t;u_0(z_2))$$

$$+ e^{ik_F(\tilde{z}(z_1)+\tilde{z}(z_2))-i\pi(N-1)} C_a(-u_0(z_1),t;u_0(z_2)) + e^{-ik_F(\tilde{z}(z_1)+\tilde{z}(z_2))+i\pi(N-1)} C_a(u_0(z_1),t;-u_0(z_2)) \Big\}.$$

Note the similarity with the representation of the one-particle correlation function for OBC in [25, 33]. However, the argument in the operators is non-linearly related to the spatial positions due to the harmonic trap. In order to apply the Wick theorem

$$\langle e^{\hat{A}}\, e^{\hat{B}}\rangle = \exp\left(\langle\hat{A}\hat{B}\rangle + \frac{1}{2}\langle\hat{A}^2\rangle + \frac{1}{2}\langle\hat{B}^2\rangle\right), \tag{A.12}$$

the operators $\hat{A}$ and $\hat{B}$ must be homogeneous linear combinations of $\hat{f}$ and $\hat{f}^+$. We call the corresponding part in equation (15) $\hat{\Phi}_f$, hence $\hat{\phi}_{odd} = \hat{\Phi}_f - b$, and get

$$C_a(u,t;v) = e^{-i(N-1)(u-v)+i\mu_N t+ib(u)-ib(v)}\,\langle e^{-i\hat{\Phi}_f(u,t)+2\pi i\hat{\Theta}(u,t)}\, e^{i\hat{\Phi}_f(v)-2\pi i\hat{\Theta}(v)}\rangle. \tag{A.13}$$

Thus $\hat{A} \equiv -i\hat{\Phi}_f(u,t) + 2\pi i\hat{\Theta}(u,t) = \hat{A}(u,t), \quad \hat{B} \equiv i\hat{\Phi}_f(v) - 2\pi i\hat{\Theta}(v) = -\hat{A}(v,0)$.

We begin with the case of zero temperature when only $\langle\hat{f}_m\hat{f}_n^+\rangle_0 = \delta_{m,n}$ survives. Then simple but lengthy calculations using equation (11) and $1/K_m \to 1 + (1 - K)\exp(-mr)/K$ and the coupling constants

$$\alpha_0 = \frac{1}{4}\left[\frac{1}{K} - K\right], \quad \gamma_0 = \frac{(1-K)^2}{4K}, \tag{A.14}$$

give

$$\exp\left(\frac{1}{2}\langle\hat{A}^2(u)\rangle_0\right) = \sqrt{\eta}\, r^{\gamma_0}\left\{\mathcal{D}(2u+ir)\mathcal{D}(-2u+ir)\right\}^{-\alpha_0/2}, \tag{A.15}$$

with $\mathcal{D}(s) \equiv 1/(1 - \exp(is))$. The correlation function $\langle\hat{A}(u,t)\hat{B}(v)\rangle_0 = -\langle\hat{A}(u,t)\hat{A}(v,0)\rangle_0$ turns out to be

$$\begin{aligned}\langle\hat{A}(u,t)\hat{B}(v)\rangle_0 &= \ln\mathcal{D}(u-v-\epsilon t+i\eta) \\ &+\gamma_0\left\{\ln\mathcal{D}(u-v-\epsilon t-ir) + \ln\mathcal{D}(v-u-\epsilon t+ir)\right\} \\ &+\alpha_0[\ln\mathcal{D}(u+v-\epsilon t+ir) + \ln\mathcal{D}(-u-v-\epsilon t+ir)].\end{aligned} \tag{A.16}$$

The final zero temperature result is

$$C_a(u,t;v) = \langle\hat{\psi}_a^+(u,t)\hat{\psi}_a(v)\rangle_0 = e^{-i(N-1)(u-v)+i\mu_N t+ib(u)-ib(v)}\, r^{2\gamma_0} \tag{A.17}$$

$$\times\frac{\left[\mathcal{D}(u-v-\epsilon t+ir)\,\mathcal{D}(v-u-\epsilon t+ir)\right]^{\gamma_0}}{1-e^{-\eta+i(u-v-\epsilon t)}}\left\{\frac{\mathcal{D}(u+v-\epsilon t+ir)\,\mathcal{D}(-u-v-\epsilon t+ir)}{|\mathcal{D}(2u+ir)\mathcal{D}(2v+ir)|}\right\}^{\alpha_0}.$$

Note that in contrast to [25] two small parameters appear in equation (A.17), the positive infinitesimal η and r which is finite in the finite system. In addition, the one-particle operators in the Hamiltonian lead to additional phase factors which can be expressed as: $e^{i(b(u)-b(v))} = \{\mathcal{D}(2u+ir)\,\mathcal{D}(-2v+ir)/(\mathcal{D}(-2u+ir)\,\mathcal{D}(2v+ir))\}^{\kappa_0}$.

The main difference of equation (A.17) to the formula (25) in [25] is the appearance of the auxiliary variable u (or $u_0(z)$) in place of the position. Otherwise, our result (A.17) is in direct correspondence to that formula. As a consequence, the finite temperature calculation of [25] can be adopted to obtain the result for the canonical ensemble by making the substitution

$$\mathcal{D}(s) = \frac{1}{1-e^{is}} \rightarrow \tilde{\mathcal{D}}(s) = \mathcal{D}(s)/\Pi_{k=1}\left[1+\left(\frac{\sin(s/2)}{\sinh(k\beta\epsilon/2)}\right)^2\right], \tag{A.18}$$

with $\beta^{-1} = k_B T$. We will not pursue the finite temperature case. In the same way, we find

$$\langle\hat{\psi}_a(u,t)\hat{\psi}_a^+(v)\rangle_0 = e^{iN(u-v)-i\mu_{N+1}t-ib(u)+ib(v)}\, r^{2\gamma_0} \tag{A.19}$$

$$\times\frac{[\mathcal{D}(u-v-\epsilon t+ir)\,\mathcal{D}(v-u-\epsilon t+ir)]^{\gamma_0}}{1-e^{-\eta+i(u-v-\epsilon t)}}\left\{\frac{\mathcal{D}(u+v-\epsilon t+ir)\,\mathcal{D}(-u-v-\epsilon t+ir)}{|\mathcal{D}(2u+ir)\mathcal{D}(2v+ir)|}\right\}^{\alpha_0}.$$

Appendix B: Backscattering between the two components

In the approximation $V_{c\perp}(|m|) \rightarrow V_{c\perp}$, the associated two-particle operator in second quantization is

$$\hat{V}_\perp = \frac{1}{2}V_{c\perp}\sum_\sigma\int_{-\pi}^{\pi}\frac{dv}{2\pi}\hat{\psi}_{a-\sigma}^+(v)\hat{\psi}_{a\sigma}^+(-v)\hat{\psi}_{a\sigma}(v)\hat{\psi}_{a-\sigma}(-v). \tag{B.1}$$

Using

$$\hat{\psi}_{a\sigma}^+(v) = e^{-i(\hat{N}_\sigma-1)v}\,\hat{U}_\sigma^+\frac{1}{\sqrt{\eta}}e^{-i\hat{\phi}_\sigma^+(v)-i\hat{\phi}_\sigma(v)} \tag{B.2}$$

together with equation (4), and utilizing $[\hat{U}_\sigma, \hat{N}_{\sigma'}] = \delta_{\sigma,\sigma'}\hat{U}_\sigma$, we obtain the following phase representation of $\hat{V}_\perp$

$$\hat{V}_\perp = \frac{V_{c\perp}}{4\pi\eta^2}\sum_\sigma\int_{-\pi}^{\pi}du\exp[2iu(\hat{N}_\sigma-\hat{N}_{-\sigma})]\exp[2i(\hat{\varphi}_{\sigma,odd}(u)-\hat{\varphi}_{-\sigma,odd}(u))], \tag{B.3}$$

with new phase operators,

$$\hat{\varphi}_{\sigma,odd}(u) = \sum_\nu\sum_{m=1}^{\infty}\sigma^{\frac{1-\nu}{2}}\sqrt{\frac{e^{-m\eta}}{2m}}\,\sin(mu)\,\left(\hat{d}_{m\nu}+\hat{d}_{m\nu}^+\right). \tag{B.4}$$

Because of $\hat{\varphi}_{\sigma,odd}(u)-\hat{\varphi}_{-\sigma,odd}(u) \equiv \sigma\sqrt{2}(\hat{\Phi}_{-1}-u\hat{N}_{\nu=-1})$ the simpler form

$$\hat{V}_\perp = \frac{V_{c\perp}}{2\pi\eta^2}\int_{-\pi}^{\pi}du\cos\left[\sqrt{8}\hat{\Phi}_{-1}(u)\right] \tag{B.5}$$

is obtained. As expected, only the composition part is affected by backscattering between components.

Finally, it is noted that that normal ordered form of equation (B.5) is

$$\hat{V}_\perp = V_{c\perp}\int_{-\pi}^{\pi}\frac{du}{2\pi}:\frac{\cos\left[\sqrt{8}\hat{\Phi}_{-1}(u)\right]}{1+e^{-2\eta}-e^{-\eta}\cos(2u)}:. \tag{B.6}$$

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